

# RIBBON HOPF ALGEBRAS FROM GROUP CHARACTER RINGS

BERTFRIED FAUSER, PETER D. JARVIS, AND RONALD C. KING

**ABSTRACT.** We study the diagram alphabet of knot moves associated with the character rings of certain matrix groups. The primary object is the Hopf algebra **Char-GL** of characters of the finite dimensional polynomial representations of the complex group  $GL(n)$  in the inductive limit, realised as the ring of symmetric functions  $\Lambda(X)$  on countably many variables  $X = \{x_1, x_2, \dots\}$ . Isomorphic as spaces are the character rings **Char-O** and **Char-Sp** of the classical matrix subgroups of  $GL(n)$ , the orthogonal and symplectic groups. We also analyse the formal character rings **Char-H $_{\pi}$**  of algebraic subgroups of  $GL(n)$ , comprised of matrix transformations leaving invariant a fixed but arbitrary tensor of Young symmetry type  $\pi$ , which have been introduced in [5] (these include the orthogonal and symplectic groups as special cases). The set of tangle diagrams encoding manipulations of the group and subgroup characters has many elements deriving from products, coproducts, units and counits as well as different types of branching operators. From these elements we assemble for each  $\pi$  a crossing tangle which satisfies the braid relation and which is nontrivial, in spite of the commutative and co-commutative setting. We identify structural elements and verify the axioms to establish that each **Char-H $_{\pi}$**  ring is a ribbon Hopf algebra. The corresponding knot invariant operators are rather weak, giving merely a measure of the writhe.

## 1. INTRODUCTION

In this article we develop a new approach to realisations of the braid group, and indirectly to knot and link invariants. In standard diagrammatic treatments, lines are decorated with linear spaces, typically modules of a suitably deformed algebra, and crossings encode the action of appropriate operators or  $R$ -matrices representing braid generators. By contrast, our lines in tangle diagrams are decorated with group characters, which do not entail any deformation of the groups. Instead, the ‘deformation’ which gives rise to nontrivial braidings, is combinatorially generated at the level of the product rule for the group characters themselves.

By way of introduction we now elaborate on these statements in order to motivate and explain the strategy underlying our work, and to clarify our claims. We shall not dwell on the above-mentioned deformed algebra approach, which has been well-studied in the literature and is covered in many standard texts [23, 9, 14]. However, we shall make central use of graphical calculus, and we pause here to explain briefly the origins of its use in knot theory. Then we indicate its extension beyond the knot alphabet, to tangle elements arising from the Hopf structure of the character rings, which we shall need for character manipulations. Indeed, the paper as a whole can be seen as a description of how the extended tangle alphabet can be used to assemble a useable graphical calculus for knots and links. The present introduction can also be read as an informal outline of the paper. A formal resumé is also provided at the end of this section.

---

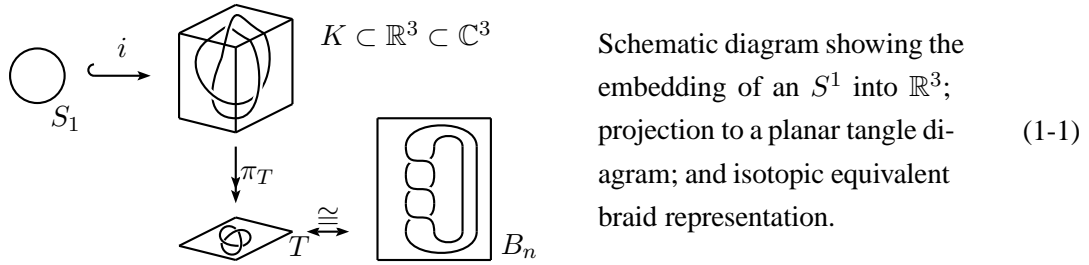
*Date:* June 12, 2011.

*2000 Mathematics Subject Classification.* Primary 16W30; Secondary 05E05; 11E57; 43A40.

*Key words and phrases.* plethysm,  $\lambda$ -rings, analytic continuation, algebraic combinatorics.

This research was supported through the programme “Research in Pairs” by the Mathematisches Forschungsinstitut Oberwolfach in 2010.

Let  $S_1$  be a circle (or let  $S_1^{\times k}$  be a direct product of  $k$ -circles in case of a link on  $k$  strands). A *knot* is a tame nonsingular embedding of the circle into  $\mathbb{R}^3$ . This means that the embedded curve is smooth (no cusps with bounded extrinsic curvature) and does not have self-crossings, hence does not have singular points. By considering projections  $\pi_T : K \rightarrow T$  of the knot into the plane, we obtain *tangles* which keep the over and under information. It is known that there is an isomorphism from knot isotopy classes to isotopy classes of tangles [14], preserving all topological information about the knot. Finally, we fix a particular projection, as a representative of the tangle isotopy. Now by the fundamental Alexander theorem, every knot can be deformed via an isotopy into a closed braid. The closed braids obtained by projection index the isotopy classes of knots, and the braid group representations become central in studying knots and their invariants, as illustrated schematically in (eqn: 1-1).



The ‘diagrammar’ of knot moves arises from the way in which *sliced tangles* are made from the tangle diagrams. For tame knots, it is possible under isotopy to distribute horizontal cuts such that at most one non-trivial, non-identity operation occupies each horizontal strip or slice. Thus, abstractly, knot or link tangles are generated using a realization of a knot *alphabet* of basic letters<sup>1</sup>:

$$\text{knot-alphabet} = \left\{ \left| \begin{array}{c} | \\ \cap \\ \cup \\ \times \\ \times \end{array} \right| \text{relations} \right\}, \quad (1-2)$$

where the ‘relations’ represent equivalence between certain words in this alphabet, implementing the isotopy equivalence of different knot projections represented by these tangles. The formal use of these elements in defining knot invariants will be dealt with in the body of the paper (§ 4).

In order to motivate the extended diagrammatic tangle alphabet which we require, and its relation to the knot alphabet, we briefly introduce the mathematical setting in which we work. Full details are elaborated in §3.2, §4, and §4.1 below.

Consider the ring of characters of finite-dimensional, polynomial irreducible representations of the complex matrix group  $GL(n)$ . It is well known that in the inductive limit  $n \rightarrow \infty$ , this ring  $\text{Char-GL}$  is isomorphic to the ring  $\Lambda(X)$  of symmetric polynomials in countably many variables  $X = \{x_1, x_2, x_3, \dots\}$  (see §2.1 for notation). As will be explained in detail in §2.1,  $\text{Char-GL}$  is a Hopf algebra. Lines in tangle diagrams are labelled by elements of the algebra (for example one may use the famous Schur- or  $S$ -functions as a basis). Both the algebra (pointwise multiplication) and bialgebra (comultiplication) structures require specific relations amongst combinations of diagrammatic elements. Including certain other types of canonical structural elements, the tangle alphabet

<sup>1</sup>We read tangle diagrams from top to bottom using the ‘pessimistic arrow of time’ [15], also our crossing is left-handed, so we are left-handed pessimists. Some literature features the right-handed optimist reading, so care is needed comparing images.

then contains the following (possibly oriented) types of elements:

$$\begin{aligned} \text{GL-alphabet} &= \left\{ \begin{array}{c} |; \circ; \circ; \cap; \cup; \text{S}; \cap; \cup; \times; \times \end{array} \mid \text{relations} \right\}, \\ &= \left\{ \text{Id}; \eta; \epsilon; \Delta; m; \text{S}; C[X; Y]; \langle - \mid - \rangle; c_{U,V}; c_{U,V}^{-1} \mid \text{relations} \right\} \quad (1-3) \end{aligned}$$

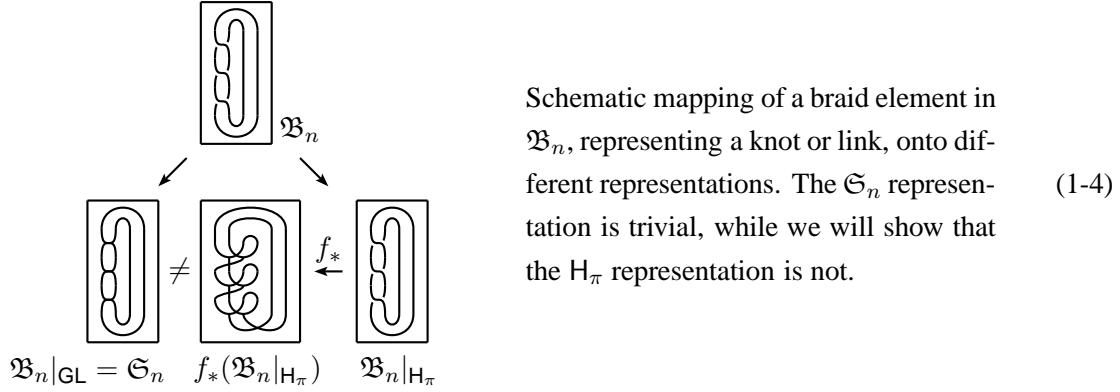
where the ‘relations’ code for associativity and units for the products and other structural compatibilities demanded by the Hopf algebra axioms. Some of these symbols will need to acquire additional attributes such as orientation; their full descriptions will of course be developed in the body of the paper.

Beyond the ambient ring **Char-GL** of symmetric functions we also require the character rings of certain algebraic matrix subgroups  $H_\pi(n)$  of  $GL(n)$ , defined as complex matrices preserving a fixed complex tensor of Young symmetry type  $\pi$  [5, 6, 7]. The rings **Char- $H_\pi$**  are isomorphic to **Char-GL** as linear spaces, and have diagrammatic elements derived from the **GL-alphabet** but subject to their own relations. Using these elements as building blocks, we can assemble composite 2-2 tangles, or crossings, which are nontrivial, and which satisfy the braid relations (necessarily so, as a result of the Hopf structure). In fact by identifying certain universal algebraic elements, and appropriately completing the alphabet of knot moves, we prove in §4.1 the main theorem of this paper (Theorem 4.16), that each **Char- $H_\pi$**  ring has the structure of a ribbon Hopf algebra [23, 8] and so can be used to represent knots.

The emergence of knot and link representations in a commutative, co-commutative setting is a striking result from the viewpoint of standard approaches using deformed algebras and braid operators to represent braid generators. As mentioned, we deal with standard matrix groups. However, from a Hopf algebraic perspective, there is indeed a deformation [20, 19], because of the changed character product rule (the  $\pi$ -Newell-Littlewood rule [5], which differs from pointwise multiplication), enjoyed by symmetric functions as elements of **Char- $H_\pi$**  rather than as elements of **Char-GL**. This deformation is combinatorially labelled by each distinct symmetry type represented by the partition  $\pi$ , rather than being labelled by a parameter.

This situation can be described dually in the following way. Recall that the symmetric group  $\mathfrak{S}_n$  plays the role of the centraliser algebra of the general linear group in tensor products. However,  $\mathfrak{S}_n$  is replaced by a different algebra for matrix subgroups of the general linear group (for example, it is replaced by the Brauer algebra for the orthogonal and symplectic groups). Labelling knot tangle diagrams or closed braids with group characters, the conversion of an element of the braid group  $\mathfrak{B}_n$  to an operation on characters in **Char-GL** can schematically be viewed as passing from  $\mathfrak{B}_n$  to  $\mathfrak{S}_n$ , whereby all crossing information is lost, with the replacement of braid moves merely by transpositions. However, the labelling with characters in **Char- $H_\pi$**  entails operations controlled by the appropriate centralizer algebra. Working in **Char-GL**, we arrive at a tangle which is a pull-back of the isomorphism between the underlying spaces, which is no longer equivalent to the directly-derived,

degenerate tangle, and hence may retain topological information (eqn: 1-4).



The remainder of the paper is organised as follows. In §2.1 we introduce the character ring **Char-GL** of finite dimensional polynomial representations of the general linear group, in the guise of the ring of symmetric functions  $\Lambda(X)$  of symmetric polynomials in an alphabet  $X$  of countably many variables. The Hopf algebra structure of  $\Lambda(X)$  is introduced and explained via the basic tangle diagram toolkit of §2.2. In §3 we recall the results of [5] and explain the group character branching rules needed to define **Char- $\mathcal{H}_\pi$** , and the modified  $\pi$ -Newell Littlewood product rule (§§3.1, 3.2). Associated with these products are 2-2 tangles, which are given diagrammatically in §3.3 and shown to satisfy the braid relation. From the point of view of knots (§4), these 2-2 tangles or crossings are the first part of the knot alphabet, the braid relations themselves being equivalent to one of the standard Reidemeister moves of knot isotopy. This connection is made in §4.1, and here the remaining elements of the knot alphabet are defined. This leads to the main result of the paper:

**Theorem 4.16:** *The character ring  $\Lambda = \text{Char-}\mathcal{H}_\pi$  is a ribbon Hopf algebra.*

For each partition  $\pi$  the space  $\Lambda \cong \text{Char-}\mathcal{H}_\pi$  equipped with the braid operators  $c^\pi$ ,  $(c^\pi)^{-1} =: \bar{c}^\pi$  together with the morphisms  $b$ ,  $d$ ,  $\bar{b}_\pi$ ,  $\bar{d}_\pi$  and the canonical writhe element  $Q_\pi$ , is a ribbon Hopf algebra.

With these ingredients at hand, in § 4.2 we identify ‘knot invariants’ in this formalism. These are formed in the standard way by cutting and opening one or more strands of the closed braid representation of the knot or link, and so are formally ring endomorphisms, or operators on characters. Their diagrammatic evaluation is followed through systematically, with the result that with our present toolkit, they turn out to be rather weak invariants. In the case of a simple knot, they merely report a measure of the writhe of the knot projection; for a link however, the invariants return the writhe of the individual components, as well as their mutual linking numbers. While this is but a small return for the large investment in mathematical structure leading to these final calculations, the underlying strategy is robust, and sheds original light on the background setting of algebraic and combinatorial aspects of representation theory [1].

Finally we want to emphasise that the novelty of our approach lies not in the rather trivial knot invariants presented here, but in the fact that we use group-subgroup branchings, and that our method works in the category of algebraic GL-subgroups, which is more general than that of semisimple Lie groups or algebras. There are several obvious extensions which need to be pursued, with a possibility of gaining better invariants. Among them are:  $q$ -deformations using Hall-Littlewood or Macdonald functions and the like, generalization to noncommutative symmetric functions, lifting our method from characters to representation modules, analyzing a Drinfeld double of our Hopf algebra, using graded modules.

## ACKNOWLEDGEMENT

This work is the result of a collaboration over several years following on from our paper [5] with the late Brian G Wybourne. BF gratefully acknowledges the Alexander von Humboldt Stiftung for *sur place* travel grants to visit the School of Mathematics and Physics, University of Tasmania, and the University of Tasmania for an honorary Research Associate appointment. Likewise PDJ acknowledges longstanding support from the Alexander von Humboldt Foundation, in particular for visits to the Max Planck Institute for Mathematics in the Sciences, Leipzig. PDJ also acknowledges the Australian American Fulbright Foundation for the award of a senior Fulbright scholarship. RCK acknowledges support for part of this research from the Leverhulme Foundation in the form of an Emeritus Fellowship. Part of this work was presented at AUSTMS 2009, and BF thanks Ruibin Zhang for support. We thank the DAAD for financial support allowing a visit of RCK and PDJ to Erlangen, and we are grateful for financial support from the Emmy-Noether Zentrum für Algebra, at the University of Erlangen. The authors thank these institutions for hospitality during our collaborative visits. Part of this work was done under a ‘Research in Pairs’ grant from the Mathematisches Forschungsinstitut Oberwolfach, and the authors also wish to record their appreciation of this award and the hospitality extended to them in Oberwolfach. Finally, this work could not have been completed without the generous support of the Quantum Computing Group, Department of Computer Science, University of Oxford, for hosting a research visit.

## 2. Char-GL AND TANGLE DIAGRAMS

**2.1. The ring of symmetric functions  $\Lambda(X)$ .** We consider characters of finite-dimensional polynomial (tensor) representations of the complex group  $GL(n)$  of  $n \times n$  nonsingular matrices, extended to a ring over  $\mathbb{Z}$  including formal subtraction as well as addition and multiplication of characters. In the inductive limit  $n \rightarrow \infty$  this object Char-GL is isomorphic to the ring of symmetric functions  $\Lambda(X)$  on an alphabet  $X$  of countably many variables  $\{x_1, x_2, x_3, \dots\}$ .  $\Lambda(X)$  has a canonical basis involving irreducible  $GL(n)$  characters, the Schur or  $S$ -functions  $\{s_\lambda\}_\lambda$  where  $\lambda$  is an integer partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ , with  $\ell(\lambda) = \ell$  the number of (positive) parts, and  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$  the weight of the partition, written  $\lambda \vdash |\lambda|$ . We follow [12] to which we refer for details; further combinatorial and notational aspects will be introduced as required. Below we turn to the attributes of  $\Lambda(X)$  which play a crucial role in our development, namely its algebraic and Hopf-algebraic structures [4]. Where no ambiguity arises,  $\Lambda(X)$  will occasionally be referred to simply as  $\Lambda$  in what follows.

**2.2. Tangle diagram tool kit.** We wish to describe the algebraic properties of  $\Lambda(X)$  in a pictorial way using diagrams. A tangle diagram is a graph or decorated graph which represents an algebraic statement. It contains zero or more pendant ‘input’ edges, nodes describing algebraic operations on whatever is being carried by the edges, and finally pendant ‘output’ edges (see for example [10, 3]). In our convention algebraic operations develop from top to bottom; directedness of the edges themselves has a different significance, as we shall see. Here we illustrate the method while introducing step-by-step the algebraic properties of  $\Lambda(X)$ .

In the simplest case of a single line or 1-1 tangle, the edge label (say  $s_\lambda$ ) is unchanged, so we have the identity map:

$$\begin{array}{c} | \\ \Leftrightarrow \end{array} \quad \text{Id}(s_\lambda) = s_\lambda \quad (\text{identity}). \quad (2-5)$$

By contrast the unit  $\eta : \mathbb{Z} \rightarrow \Lambda$  is an injection map from the underling ring, say  $\mathbb{Z}$ , into the ring of symmetric functions and is a 0-1 tangle:

$$\begin{array}{c} \circ \\ | \end{array} \Leftrightarrow \eta(1) = s_0 \quad (\text{unit}). \quad (2-6)$$

These examples illustrate the convention that diagrams with unlabelled edges specify maps, while edge labelling corresponds to specifying the action of maps on elements. 0- $n$  tangles are injection maps and so have no input line, the default action being on the scalar unit, 1; conversely  $n$ -0 tangles are scalar-valued, and so have no output line.

The symmetric function outer product is simply pointwise multiplication, and so is a 2-1 tangle,  $m : \Lambda \otimes \Lambda \rightarrow \Lambda$ :

$$\begin{array}{c} | \\ \cup \end{array} \Leftrightarrow m(s_\lambda \otimes s_\mu) = s_\lambda \cdot s_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} s_{\nu} \quad (\text{outer product}). \quad (2-7)$$

The coefficients  $c_{\lambda, \mu}^{\nu}$  in the  $S$ -function basis are the famous Littlewood-Richardson coefficients giving the multiplicity of irreducible parts in the decomposition of a tensor product. The dual operation, a 1-2 tangle, is the outer coproduct map  $\Delta : \Lambda \rightarrow \Lambda \otimes \Lambda$ :

$$\begin{array}{c} | \\ \cap \end{array} \Leftrightarrow \Delta(s_\lambda) = \sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} s_{\mu} \otimes s_{\nu} \quad (\text{outer coproduct}). \quad (2-8)$$

which can be thought of as associating to a given symmetric function, a sum of bilinears in symmetric functions from two distinct alphabets. A streamlined notation for this sum is the Sweedler convention [19],  $\Delta(s_\lambda) = \sum s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}}$ . These diagrams show also that the juxtaposition of pendant edges is labelled by an element of the tensor product of copies of  $\Lambda$ , in this case  $\Lambda \otimes \Lambda$ , or  $\otimes^n \Lambda$  for a tangle with  $n$  input edges – the most general tangle diagram thus represents an algebraic statement in the tensor algebra  $T(\Lambda)$ .

Accompanying the outer coproduct is the counit, a 1-0 tangle  $\varepsilon : \Lambda \rightarrow \mathbb{Z}$ , mapping a symmetric function to a ring element (linear form):

$$\begin{array}{c} | \\ \circ \end{array} \Leftrightarrow \varepsilon(s_\lambda) = \delta_{\lambda, (0)} \quad (\text{counit}).$$

An important adjunct to the above is another operation, that of  $S$ -function *skew* which should be thought of as an endomorphism. This is the formal dual of multiplication with respect to the Schur-Hall scalar product,

$$\begin{array}{c} \lambda \\ \circ \\ \cup \end{array} \Leftrightarrow s_{\lambda}^{\perp}(s_{\mu}) = \sum \langle s_{\lambda} | s_{\mu}^{(1)} \rangle s_{\mu}^{(2)} \quad (\text{skew by } s_{\lambda}).$$

Defining alternatively  $\langle D(s_{\lambda}) s_{\mu} | s_{\nu} \rangle = \langle s_{\mu} | s_{\lambda} s_{\nu} \rangle$ , we have  $D(s_{\lambda}) = s_{\lambda}^{\perp}$  and the explicit form  $s_{\lambda}^{\perp}(s_{\mu}) = \sum_{\alpha} c_{\lambda, \alpha}^{\mu} s_{\alpha}$ . Because of the similarity to ‘division’, the skew is often denoted  $s_{\mu/\lambda} := s_{\lambda}^{\perp}(s_{\mu})$ .

A further structural element is the antipode map,

$$\begin{array}{c} s \\ | \\ \circ \end{array} \Leftrightarrow S(s_{\lambda}) = (-1)^{|\lambda|} s_{\lambda'} \quad (\text{antipode}), \quad (2-9)$$

related to the  $\omega$ -involution defined in [12], which serves to illustrate a final convention, that endomorphisms (linear operators) are designated by in-line symbols, which still require edge labelling for their

action on elements to be specified. Here  $\lambda'$  denotes the transposed partition, formed by interchanging rows and columns of  $\lambda$ .

The setting in linear algebra implied by the elements described so far is formalised by noting further that  $\Lambda(X)$  is given the structure of a Hilbert space, with orthonormal basis  $\{s_\lambda\}_\lambda$ , by the famous Schur-Hall scalar product

$$\bigcup \Leftrightarrow \langle s_\mu | s_\nu \rangle = \delta_{\mu,\nu} \quad (\text{Schur-Hall scalar product}). \quad (2-10)$$

This has a dual which injects the canonical element given by the sum over paired basis vectors,

$$\bigcap \Leftrightarrow \sum_\lambda s_\lambda \otimes s_\lambda \quad (\text{Cauchy kernel } C[X, Y]). \quad (2-11)$$

Writing this in another way using two alphabets, we have the famous Cauchy identity [12],

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j} \frac{1}{(1 - x_i y_j)} = C[X, Y], \quad (2-12)$$

whereas the dual Cauchy kernel is given by either of the following forms,

$$\begin{aligned} \bigcap^s &\Leftrightarrow \text{Id} \otimes S \circ \sum_\lambda s_\lambda \otimes s_\lambda = \sum_\lambda (-1)^{|\lambda|} s_\lambda \otimes s_{\lambda'}, & (\text{Cauchy-Binet kernel}), \\ s \bigcap &\Leftrightarrow S \otimes \text{Id} \circ \sum_\lambda s_\lambda \otimes s_\lambda = \sum_\lambda (-1)^{|\lambda|} s_{\lambda'} \otimes s_\lambda, & (\text{Cauchy-Binet kernel}), \end{aligned}$$

with 
$$\sum_\lambda (-1)^{|\lambda|} s_{\lambda'}(X) s_\lambda(Y) = \sum_\lambda (-1)^{|\lambda|} s_\lambda(X) s_{\lambda'}(Y) = \prod_{i,j} (1 - x_i y_j), \quad (2-13)$$

the latter being the Cauchy-Binet formula. A final example is the 2-2 tangle representing the transposition or switch operator, which is simply

$$\times \Leftrightarrow \text{SW}(s_\mu \otimes s_\nu) = s_\nu \otimes s_\mu \quad (\text{switch}). \quad (2-14)$$

This tangle is planar and does not record any over or under information.

**2.3. Char-GL as a Hopf algebra.** We now give the properties satisfied by  $\Lambda \cong \text{Char-GL}$  as a Hopf algebra, in the form of tangle diagrams. Firstly, the following statements of the associativity of multiplication  $\mathfrak{m}$  and comultiplication  $\Delta$  are self-evident:

$$\begin{aligned} \bigcup \approx \bigcup \approx \bigcup, & \quad \bigcap \approx \bigcap \approx \bigcap. \end{aligned} \quad (2-15)$$

It is useful to note that as a consequence, associativity and co-associativity iterate to  $n-1$  and  $1-n$  tangles which are independent of the bracketing order, giving, effectively, diagrammatical elements of star type. Moreover, the algebra and coalgebra are unital and counital, respectively:

$$\begin{aligned} \bigcup \approx \bigcup \approx |, & \quad \bigcap \approx \bigcap \approx |, \end{aligned} \quad (2-16)$$

Also, in the particular case of  $\Lambda$ , multiplication  $m$  is commutative and comultiplication  $\Delta$  is cocommutative:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array}, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} \diagdown \diagdown \\ \diagup \diagup \end{array}, \quad (2-17)$$

The bialgebra property asserts

$$\begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \approx \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \Leftrightarrow \Delta(fg) = \Delta(f) \cdot \Delta(g), \quad (2-18)$$

where  $f$  and  $g$  are any symmetric functions. The operator representing the tangle giving the right-hand side is  $m \otimes m \circ 1 \otimes \text{sw} \otimes 1 \circ \Delta \otimes \Delta(f \otimes g)$ . If we write generically [19]  $\Delta f = \sum f_{(1)} \otimes f_{(2)}$ , then the axiom reads

$$\Delta(f \cdot g) = \sum f_{(1)} \cdot g_{(1)} \otimes \sum f_{(2)} \cdot g_{(2)}. \quad (2-19)$$

Finally, augmenting the bialgebra property, the antipode map satisfies the compatibility conditions

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \circ \text{S} \approx \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \circ \text{S} \approx \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \quad (2-20)$$

which state that the (idempotent) antipode is the convolutive inverse of the identity map. Finally we have

**Theorem 2.1: The outer Hopf algebra of symmetric functions:** The sextuple  $(\Lambda, m, \Delta, \mathbf{S}, \eta, \varepsilon)$  with the above bialgebra, (co)unit and antipode axioms, is a commutative, co-commutative Hopf algebra, the outer Hopf algebra of symmetric functions. ■

**Proof:** see for example [22, 21, 4]. □

To end this section, the following display exhibits several tangle relations expressing algebraic identities that are easily checked in the Char-GL case, where the under and over crossings are identical.

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} | \\ | \end{array} \approx \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array}$$

**R0:** The zeroth or topological Reidemeister move, also the closure of the tangle category. (2-21)

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} | \\ | \end{array} \approx \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$$

**R1:** The first Reidemeister move. (2-22)

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} | \\ | \end{array} \approx \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$$

**R2:** The second Reidemeister move. (2-23)

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \approx \begin{array}{c} | \\ | \end{array} \approx \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$$

**R3:** The third Reidemeister move, also quantum Yang-Baxter equation. (2-24)

The first two relations are ‘straightening rules’ which allow loops to be removed; the second two statements mean that the switch  $\text{sw}$  has all the properties of an elementary transposition operator in the symmetric group, if the permutations (on  $n$  objects say) are realised as  $n$ - $n$  tangles representing corresponding line shuffles. Thus the third diagram asserts that elementary transpositions are involutive, while the fourth diagram gives the standard exchange relation for neighbouring transpositions in



the presentation of the symmetric group:

$$\mathfrak{S}_n = \langle s_i, i \in \{1, 2, \dots, n\} : s_i^2 = \text{Id}, s_i s_j = s_j s_i, |i - j| \geq 2, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \quad (2-25)$$

under the morphism  $\mathfrak{B}_n \rightarrow \mathfrak{S}_n$  under which each braid generator  $b_i$  maps to the corresponding transposition  $s_i$ .

These relations serve as a template for the idea that tangle diagrams are subject to rearrangement by simplification rules. This theme will be developed with the subgroup Hopf algebras to be introduced in later sections, and will of course be replaced by statements enabling manipulations with nontrivial braid operators  $c_{i,j}$ , when the correct identifications are given.

### 3. THE CHARACTER RINGS $\text{Char-H}_\pi$

For certain matrix subgroups of  $GL(n)$ , the characters can be handled by an extension of the symmetric function methods in  $\Lambda(X)$  used for  $\text{Char-GL}$ , and these form the subject of the present section (see §3.1 below). We frequently illustrate the results by the special case of the *classical* matrix subgroups, the orthogonal and symplectic groups  $O(n)$  and  $Sp(n)$ , respectively (with  $n$  even in the latter case). The general class of subgroups  $H_\pi(n)$  which we consider includes not only, for example, odd-dimensional symplectic groups [16], and orthogonal groups with singular metric, but generically non-semisimple matrix groups and even discrete groups [5]. However, the tangle notation for  $H_\pi(n)$  is more involved (see §3.3).

**3.1. Group branching rules.** The character rings of the orthogonal and symplectic groups can be treated with symmetric function methods [7] provided care is given to the way in which the conjugacy classes (parametrised by eigenvalues of the group matrices) are handled. In the inductive limit,  $\text{Char-O}$  and  $\text{Char-Sp}$  are, as linear spaces, isomorphic copies of  $\text{Char-GL}$ , whereas the product rule for their characters is different. The isomorphism of spaces is called the group-subgroup branching rule, and reflects at the character level the decomposition of irreducible representations, on restriction to a subgroup. The multiplication of characters (corresponding to the reduction of a tensor product) is called the Newell-Littlewood rule [13, 11]. Similarly for the groups  $H_\pi(n)$  in the inductive limit, we shall introduce the character rings  $\text{Char-H}_\pi$ . In the next subsections we give the  $\pi$ -group-subgroup branching rules, and in the following subsection we give the  $\pi$ -Newell-Littlewood product rule.

**3.1.1.  $S$ -function series and plethysms.** In order to introduce the group-subgroup branching rule, we first turn to systematics of  $S$ -function series [12]. These are formal infinite sums of symmetric functions of a specific type (often involving partitions of a particular shape if referred to the basis of  $S$ -functions) whose elements are used term-wise in multiplication and skewing operations (§2.2). More correctly, the ring  $\Lambda$  is extended to  $\Lambda[[t]]$ , with the convention that if the symmetric functions appearing in the coefficient  $[t^n]$  are graded by partition weight, then the indeterminate  $t$  is redundant and is usually omitted.

The ring  $\Lambda$  has various linear and multiplicative bases whose elements are succinctly given by generating series. The most useful are

$$\begin{aligned} M_t &= \prod_i \frac{1}{1 - tx_i} = \sum_n h_n t^n, \\ L_t &= \prod_i (1 - tx_i) = \sum_n e_n (-t)^n, \\ P_t &= -\ln M_t := \sum_n p_n t^n / n, \end{aligned}$$

$$\begin{aligned} \text{where } p_n(X) &= \sum_i x_i^n, \\ h_n(X) &= \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}, \\ e_n(X) &= \sum_{1 \leq i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \dots x_{i_n} \end{aligned}$$

are the power sum, complete and elementary symmetric functions, respectively. The complete symmetric functions are in fact  $S$ -functions for one part partitions,  $h_n \equiv s_{(n)}$ , and for partitions with each part at most 1 (Young diagrams with one column, the transpose of the one part partitions),  $e_n \equiv s_{(1^n)}$ , respectively. In general each Schur function can be written

$$s_\lambda(X) = \sum_T x^T, \quad (3-26)$$

where  $T$  is a monomial in  $x_i$  derived from the entries in semistandard tableaux  $F_\lambda$  filling the Young frame or Ferrers diagram of  $\lambda$ . For example,  $s_{(2,1)}(X) = \sum_{i \neq j} x_i^2 x_j + 2 \sum_{i < j < k} x_i x_j x_k$ . In contexts where the alphabet is unambiguous, the Littlewood convention  $s_\lambda(X) = \{\lambda\}(X)$  or simply  $s_\lambda = \{\lambda\}$  is sometimes useful for Schur functions, in order to emphasize the specific partition and if the alphabet is understood. Thus for example  $s_{(2,1)}(X)$  can be written as  $\{2, 1\}$ .

In the following we shall also be interested in cases of the operation of symmetric function *plethysm* which is defined as follows [7]. For  $\lambda$  a symmetric function let  $Y = \{x^T\}_{T \in F_\lambda}$  be the alphabet corresponding to the monomials  $x^T$ . Then for any symmetric function  $f$ , the plethysm of  $f$  by  $\lambda$  is  $f[s_\lambda](X) := f(Y)$ . In particular we can evaluate  $s_\mu[s_\lambda](X)$  in this way. Applied to the above series, for a fixed partition  $\pi$ , consider the semistandard tableaux  $T \in F_\pi$  as above. We define

$$\begin{aligned} M_\pi &\equiv M[s_\pi] = \prod_{T \in F_\pi} \frac{1}{(1 - x^T)}, & L_\pi &\equiv L[s_\pi] = \prod_{T \in F_\pi} (1 - x^T); \\ \text{thus } M_{\{2\}} &= \prod_{i \leq j} \frac{1}{(1 - x_i x_j)}, & L_{\{2\}} &= \prod_{i \leq j} (1 - x_i x_j), \\ M_{\{1,1\}} &= \prod_{i < j} \frac{1}{(1 - x_i x_j)}, & L_{\{1,1\}} &= \prod_{i < j} (1 - x_i x_j). \end{aligned}$$

Standard notation for the above is [11]  $M_{\{2\}} = D = \sum_\delta s_\delta$ ,  $L_{\{2\}} = C = \sum_\gamma (-1)^{\frac{1}{2}|\gamma|} s_\gamma$ ,  $M_{\{1,1\}} = B = \sum_\beta s_\beta$ ,  $L_{\{1,1\}} = A = \sum_\alpha (-1)^{\frac{1}{2}|\alpha|} s_\alpha$ . Here  $\{\delta\}$  is the set of partitions with each part (row) even;  $\{\gamma\}$  is the set of partitions whose principal hooks (nested, inverted L-shaped strips down the

diagonal) have the shape  $(a_i + 1, 1^{b_i})$  with  $a_i = b_i + 1$  (arm length exceeds leg length by 1);  $\beta$  is the set of partitions  $\delta'$ , and  $\alpha$  is the set of  $\gamma'$ .

For higher rank plethysms we have for example

$$\begin{aligned} M_{\{3\}} &= \prod_{i \leq j \leq k} \frac{1}{(1 - x_i x_j x_k)}, & L_{\{3\}} &= \prod_{i \leq j \leq k} (1 - x_i x_j x_k), \\ M_{\{2,1\}} &= \prod_{i \neq j} \frac{1}{(1 - x_i^2 x_j)} \prod_{i < j < k} \frac{1}{(1 - x_i x_j x_k)}, & L_{\{2,1\}} &= \prod_{i \neq j} (1 - x_i^2 x_j) \prod_{i < j < k} (1 - x_i x_j x_k). \end{aligned} \quad (3-27)$$

For  $\pi$  of rank 3 and higher, the first few terms in the series  $M_{\{3\}}, M_{\{2,1\}}, \dots$  can be evaluated explicitly, but there is no known systematic description of the patterns occurring at arbitrary degree.

**3.1.2. Group branching rules for  $\pi$ -characters.** As mentioned above, at the character ring level, the group-subgroup branching rule is a linear space isomorphism. In the case of the orthogonal and symplectic subgroups of  $GL(n)$ , in the  $S$ -function basis, the images of  $s_\lambda$  are denoted  $o_\lambda$  and  $sp_\lambda$ , respectively (Schur functions of orthogonal and symplectic type) and are irreducible characters of the respective subgroups. The branching isomorphisms are denoted  $/D$  and  $/B$ , respectively:

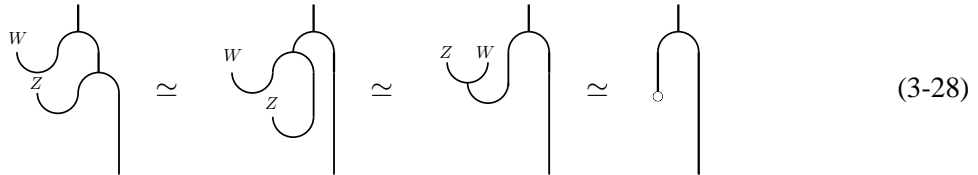
**Theorem 3.2: Group branching rules – orthogonal and symplectic groups.**

$$\begin{aligned} /D : \text{Char-GL} &\rightarrow \text{Char-O}, & s_\lambda &\mapsto o_{\lambda/D}; \\ /D^{-1} \equiv /C : \text{Char-O} &\rightarrow \text{Char-GL}, & o_\lambda &\mapsto s_{\lambda/C}; \\ /B : \text{Char-GL} &\rightarrow \text{Char-Sp}, & s_\lambda &\mapsto sp_{\lambda/B}; \\ /D^{-1} \equiv /C : \text{Char-Sp} &\rightarrow \text{Char-GL}, & sp_\lambda &\mapsto s_{\lambda/A}. \end{aligned}$$

■

**Proof:** See for example Littlewood [11]. The fact that the inverse maps are also series branchings by the inverse series is due to the distributivity law for the symmetric function skew product, namely  $f/(gh) = (f/g)/h$ , which follows trivially from the duality between skew product and outer multiplication. See also [7]. □

Tangle diagrams for the group branching laws (skewing by symmetric function series) are introduced as in § 2.2 above. For example, the following manipulations establish the inverse branchings, for any series  $W$  and  $Z$  such that  $WZ = 1$ , implementing  $(s_\lambda/W)/Z = s_\lambda/(WZ) \equiv s_\lambda$ :



Finally we turn to the general  $H_\pi$  groups and their character rings  $\text{Char-}H_\pi$ . Fix a partition  $\pi$  and a complex tensor (an element of  $\otimes^{|\pi|}(\mathbb{C}^n)$ ) of Young symmetry type  $\pi$ . Define  $H_\pi(n)$  to be the subgroup of the group  $GL(n)$  of nonsingular complex  $n \times n$  matrices which leave invariant the fixed tensor under the natural action of  $GL(n)$  induced by that on the fundamental representation  $\mathbb{C}^n$ . Clearly this group depends critically on the canonical form of the fixed tensor, and may indeed be trivial, or possibly discrete. In general, however, it will be a certain algebraic subgroup of  $GL(n)$  and – except in the case of the classical orthogonal and symplectic subgroups – will be non-semisimple (as mentioned above, including for example, symplectic groups in odd dimensions, orthogonal groups

with singular metrics). The ring  $\text{Char-H}_\pi$  (over  $\mathbb{Z}$ ) is an isomorphic copy of  $\Lambda$  consisting of formal characters of finite dimensional, in general indecomposable, representations of  $H_\pi(n)$  in the inverse limit, defined via branching maps [5]. In analogy with the orthogonal and symplectic cases, we introduce symmetric functions of type  $H_\pi$  denoted  $s_\lambda^{(\pi)}$ , defined via branching maps as follows:

**Definition 3.3: Symmetric functions of type  $H_\pi$**

$$\begin{aligned} /M_\pi : \text{Char-GL} &\rightarrow \text{Char-H}_\pi, & s_\lambda &\mapsto s_{\lambda/M_\pi}^{(\pi)}; \\ /L_\pi : \text{Char-H}_\pi &\rightarrow \text{Char-GL}, & s_\lambda^{(\pi)} &\mapsto s_{\lambda/L_\pi}. \end{aligned}$$

■

**3.2.  $\pi$ -Newell-Littlewood product and associated tangles.** We now turn to the product rule for subgroup characters in the rings  $\text{Char-H}_\pi$ , which generalises that for the orthogonal and symplectic groups,  $\text{Char-O}$  and  $\text{Char-Sp}$ . For the latter cases, the product of characters corresponds to the decomposition of a tensor product of irreducible representations, and in terms of  $S$ -functions is called the Newell-Littlewood rule,

**Corollary 3.4: Newell-Littlewood rule:**

$$o(\lambda)o(\mu) = \sum_{\alpha} o(\lambda/\alpha \cdot \mu/\alpha), \quad sp(\lambda)sp(\mu) = \sum_{\alpha} sp(\lambda/\alpha \cdot \mu/\alpha),$$

**Proof:** See Newell [13], and also Littlewood [11]. This is a special case of the  $\pi$ -Newell-Littlewood rule below (Theorem 3.5) proved in [5].

□

The generalisation to the product of characters in  $\text{Char-H}_\pi$  requires a discussion of the Hopf structure of  $s_\pi$ . For clarity of writing, we write  $s(\lambda)$  or  $\{\lambda\}$ , and  $o(\mu)$ ,  $sp(\nu)$ ,  $s^{(\pi)}(\alpha)$ , to denote Schur functions, and symmetric functions of orthogonal, symplectic and  $H_\pi$  type, in  $\Lambda$ , respectively. Where no confusion arises, enclosing braces  $\{\cdot\}$  on Schur functions can be omitted (where it is clear that expressions are evaluated in the  $S$ -function basis, as opposed to acting on general elements  $f, g, \dots$ )<sup>2</sup>. In the same vein, Schur function plethysms can be abbreviated, for example  $\alpha[\beta]$ . Let  $\{\{\pi'_{(1)}\} \otimes \{\pi'_{(2)}\}\}$  be the list *with repetition* of entries of the cut outer coproduct  $\Delta'(\{\pi\})$  of  $\{\pi\}$ , that is,

$$\begin{aligned} \Delta(\{\pi\}) - \{\pi\} \otimes 1 - 1 \otimes \{\pi\} &= \Delta'(\{\pi\}) = \sum \{\pi'_{(1)}\} \otimes \{\pi'_{(2)}\}, \\ \text{or simply} \quad \Delta(\pi) - \pi \otimes 1 - 1 \otimes \pi &= \Delta'(\pi) = \sum \pi'_{(1)} \otimes \pi'_{(2)}. \end{aligned}$$

Let the cardinality of this list be  $p = |\Delta'\pi|$ . Let  $\{\alpha_k\}_{k=1}^p$  be a set of  $p$  partitions, and denote by  $\sum_{\alpha} = \sum_{\alpha_1, \alpha_2, \dots, \alpha_p}$  the summation over all such  $p$ -tuples of partitions. We have

**Theorem 3.5: Generalised Newell-Littlewood rule:**

$$m_\pi(s^{(\pi)}(\lambda), s^{(\pi)}(\mu)) = s^{(\pi)}(\lambda) \odot s^{(\pi)}(\mu) = \sum_{\alpha_k} s^{(\pi)}(\lambda / \prod_{k=1}^p \alpha_k[\pi'_{(1)}] \cdot \mu / \prod_{k=1}^p \alpha_k[\pi'_{(2)}]),$$

**Proof:** See [5].

□

---

<sup>2</sup>Concrete instances such as  $\{2, 1\}$  still require braces to distinguish  $S$ -functions from the corresponding partitions, in this case  $(2, 1)$ .

The Newell-Littlewood rule and its  $\pi$ -generalisation can be explicitly constructed via the above decompositions, at least for concrete cases. Firstly, note that the orthogonal and symplectic groups are, by definition, matrix subgroups of  $GL(n)$  which leave invariant a bilinear form which is symmetric, or antisymmetric, respectively, so that  $\pi = \{2\}$  or  $\pi = \{1, 1\}$ . Noting that

$$\Delta(\{2\}) = \{2\} \otimes 1 + 1 \otimes \{2\} + \{1\} \otimes \{1\}, \quad \Delta(\{1, 1\}) = \{1, 1\} \otimes 1 + 1 \otimes \{1, 1\} + \{1\} \otimes \{1\},$$

we have in both cases  $\Delta'(\{\pi\}) = \{1\} \otimes \{1\}$ , so there is only one summand  $\alpha$ , and moreover the plethysm is trivial,  $\{\alpha[1]\} \equiv \{\alpha\}$  so the standard Newell-Littlewood result is recovered.

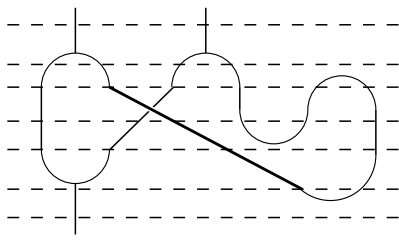
For  $|\pi| > 2$  however there is more than one part in  $\Delta'(\{\pi\})$  and so summands  $\alpha_1, \alpha_2, \dots, \alpha_p$  are required, and at least some of these will come with nontrivial plethysms if parts of  $\Delta(\{\pi\})$  are of rank 2 or larger. For example.

$$\Delta(\{3\}) = \{3\} \otimes 1 + 1 \otimes \{3\} + \{1\} \otimes \{2\} + \{2\} \otimes \{1\},$$

so that with 2 summands  $\alpha_1, \alpha_2$  or  $\alpha, \beta$  say,


$$s^{(3)}(\lambda) \odot s^{(3)}(\mu) = \sum_{\alpha, \beta} s^{(3)}(\lambda/(\alpha \cdot \beta[2]) \cdot \mu/(\alpha[2] \cdot \beta)).$$

Using tangle diagrams, these deformed products are readily illustrated. In the orthogonal and symplectic cases we have

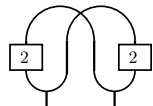


$$\begin{aligned} & \lambda \otimes \mu \\ & \lambda_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(1)} \otimes \mu_{(2)} \\ & \sum_{\alpha} \lambda_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(1)} \otimes \mu_{(2)} \otimes \alpha \otimes \alpha \\ & \Leftrightarrow \sum_{\alpha} \lambda_{(1)} \otimes \mu_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(2)} \otimes \alpha \otimes \alpha \quad (3-29) \\ & \sum_{\alpha} \lambda_{(1)} \otimes \mu_{(1)} \otimes \lambda_{(2)} \otimes \langle \mu_{(2)} | \alpha \rangle \alpha \\ & \sum_{\alpha} \lambda_{(1)} \cdot \mu_{(1)} \otimes \lambda_{(2)} \otimes \langle \mu_{(2)} | \alpha \rangle \alpha \\ & \sum_{\alpha} \lambda_{(1)} \cdot \mu_{(1)} \langle \lambda_{(2)} | \alpha \rangle \langle \mu_{(2)} | \alpha \rangle \end{aligned}$$

Summing over  $\alpha$  and using orthonormality of the  $S$ -functions, the last line becomes  $\sum \lambda_{(1)} \cdot \mu_{(1)} \langle \lambda_{(2)} | \mu_{(2)} \rangle$  or  $\sum_{\alpha} \lambda/\alpha \cdot \mu/\alpha$ , thus yielding the required form of the Newell-Littlewood rule, here denoted  $\lambda \odot \mu$ . In the general case, the Cauchy kernel is replaced by a more complicated 0-2 tangle which injects the required summands with appropriate plethysms. Calling the Cauchy kernels  $r_{\{2\}} = r_{\{1,1\}}$ , we have for  $\pi = \{3\}$  the corresponding operator  $r_{\{3\}}$ , where



$$\Leftrightarrow r_{\{2\}} = r_{\{1,1\}} :: 1 \mapsto \sum_{\alpha} \alpha \otimes \alpha, \quad (3-30)$$



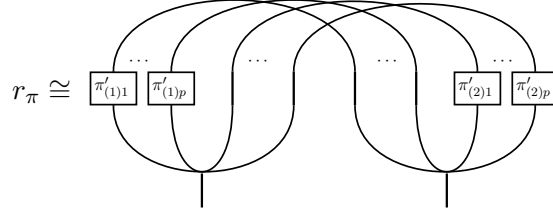
$$\Leftrightarrow r_{\{3\}} :: 1 \mapsto \sum_{\alpha, \beta} \alpha[2] \cdot \beta \otimes \alpha \cdot \beta[2]. \quad (3-31)$$

In general the 0-2 tangle  $r_{\pi}$  is a convolutive product of  $p = |\Delta'\pi|$  Cauchy kernels, whose downward lines are modified by the insertion of plethysms coming from the corresponding cut coproduct parts of  $\pi$ .

**3.3. Crossings and the braid relation.** Having identified a large class of nontrivial new product rules on  $\Lambda$  arising from the decomposition of characters of type  $H_{\pi}$  (with each partition  $\pi$  of rank  $> 2$  giving a distinct multiplication) we explore in this subsection the further ramifications of this structure. As explained above, the central feature of the deformed multiplication is that it is an appropriate convolution of ordinary multiplication, with a type of 0-2 tangle,  $r_{\pi}$  (which is itself a convolution of modified Cauchy kernels). Another object closely related to  $r_{\pi}$  is a 2-2 tangle on  $\Lambda \otimes \Lambda$ , denoted

$R_\pi$ . It is the task of this subsection to exploit the properties of  $r_\pi$  in order to investigate  $R_\pi$ . We follow [8, 23] (see also [14]). The outcome will be that  $R_\pi$  can be regarded as a new type of crossing, and indeed satisfies the quantum Yang-Baxter relation, or when composed with the plain switch  $\text{SW}$ , the braid relation. These findings will provide the starting point for exhibiting the full alphabet of knot moves (§ 4 below).

**Definiton 3.6: Generalised Cauchy kernel  $r_\pi$  and Cauchy scalar  $Q_\pi$ :**



where

$$r_\pi = \sum_{\alpha} \alpha_{(1)}^\pi \otimes \alpha_{(2)}^\pi := \sum_{\alpha} \prod_{k=1}^p \alpha_k[\pi'_{(1)}] \otimes \prod_{k=1}^p \alpha_k[\pi'_{(2)}], \quad (3-32)$$

where the boxed operators  $\pi'_{(n)}$  on downward lines signify that the appropriate  $[\pi'_{(n)}]$  plethysms are to be applied to the running variables  $\alpha_1, \alpha_2, \dots, \alpha_p$ . ■

Here and in the sequel we adopt the convenient short-hand notation  $r_\pi = \sum_{\alpha} \alpha_{(1)}^\pi \otimes \alpha_{(2)}^\pi$  for the cut co-product-derived Sweedler sums occurring in the coscalar product; in Theorem 3.5 this convention would instead lead to  $\sum_{\alpha} \cdot / \alpha_{(1)}^\pi \otimes \cdot / \alpha_{(2)}^\pi$ . In this notation the following scalar element or ‘tadpole’, regarded as a 0-1 tangle<sup>3</sup>, arises as the multiplicative closure of the bottom lines of  $r_\pi$ ,

$$Q_\pi = \sum_{\alpha} \alpha_{(1)}^\pi \cdot \alpha_{(2)}^\pi. \quad (3-33)$$

The fact that the  $\pi$ -modified Newell-Littlewood rule gives rise to an associative multiplication in  $\Lambda$  is guaranteed by the underlying isomorphism of character rings (see [5, 7] for a direct proof). Associativity for such a deformed product is equivalent to the 2-chain affiliated to  $r_\pi$  being a 2-cycle, in the appropriate Hopf algebra homology. For completeness, we give a short derivation of this equivalence.

**Lemma 3.7: The 2-chain associated to  $r_\pi$  is a 2-cycle:** ■

**Proof:** We shall not need here the full theory of Hopf algebra deformations, for which we refer to [20, 18, 17, 2, 4, 5, 7] for details. The lemma is dual to the same statement for 2-cochains and 2-cocycles, and we proof the typographical simpler dual result. For any 3 symmetric functions  $(f, g, h)$ , the 2-cocycle condition for some 2-cochain  $c : \Lambda \otimes \Lambda \rightarrow \mathbb{C}$ , reads in terms of Sweedler parts

$$\sum c(g_{(1)}, h_{(1)}) c(f, g_{(2)} h_{(2)}) = \sum c(f_{(1)}, g_{(1)}) c(f_{(2)} g_{(2)}, h). \quad (3-34)$$

In the present case, affiliated with the 2-chain  $r_\pi$  is the following 2-cochain (here denoted simply  $r$ ),

$$r(f, g) := \sum_{\alpha} \langle f | \alpha_{(1)}^\pi \rangle \langle g | \alpha_{(2)}^\pi \rangle. \quad (3-35)$$

<sup>3</sup>Note that the 0-2 and 0-1 tangles  $r_\pi$  and  $Q_\pi$  are technically homomorphisms, so could also have been defined as  $r_\pi(1)$  and  $Q_\pi(1)$ , respectively.

The circle product, whose associativity is crucial to the  $\pi$ -Newell-Littlewood theorem, then reads

$$f \odot g = \sum r(f_{(1)}, g_{(1)}) f_{(2)} g_{(2)}. \quad (3-36)$$

The expansion of  $(f \odot g) \odot h = f \odot (g \odot h)$  in Sweedler parts yields

$$\begin{aligned} \sum r(f_{(1)}, g_{(1)}) r(f_{(21)} g_{(21)}, h_{(1)}) f_{(22)} g_{(22)} h_{(2)} = \\ \sum r(g_{(1)}, h_{(1)}) r(f_{(1)}, g_{(21)} h_{(21)}) f_{(2)} g_{(22)} h_{(22)}. \end{aligned} \quad (3-37)$$

However, using coassociativity and relabelling the Sweedler sums, this becomes

$$\begin{aligned} \sum r(f_{(11)}, g_{(11)}) r(f_{(12)} g_{(12)}, h_{(1)}) f_{(2)} g_{(2)} h_{(2)} = \\ \sum r(g_{(11)}, h_{(11)}) r(f_{(1)}, g_{(12)} h_{(12)}) f_{(2)} g_{(2)} h_{(2)}, \end{aligned} \quad (3-38)$$

wherein the coefficients of  $f_{(2)} g_{(2)} h_{(2)}$  terms on each side agree, because of the 2-cocycle condition applied to the triple  $(f_{(1)}, g_{(1)}, h_{(1)})$ .  $\square$

As well as deforming the product, we can introduce an associated deformation of the coproduct by dualising,

$$\Delta_\pi(f) = \sum_\alpha f_{(1)} \cdot \prod_{k=1}^p \alpha_k[\pi'_{(1)}] \otimes f_{(2)} \cdot \prod_{k=1}^p \alpha_k[\pi'_{(2)}] \equiv R_\pi \circ \Delta(f) \quad (3-39)$$

which is as indicated the composition of the standard outer coproduct with a 2-2 tangle  $R_\pi$ ,

$$R_\pi \cong \text{tangle diagram} \quad \text{where} \quad R_\pi(f \otimes g) = \sum_\alpha f \cdot \prod_{k=1}^p \alpha_k[\pi'_{(1)}] \otimes g \cdot \prod_{k=1}^p \alpha_k[\pi'_{(2)}]. \quad (3-40)$$

We have the following

**Theorem 3.8: Co-quasitriangularity of  $\Lambda$  under  $r_\pi$ :** The co-scalar product  $r_\pi$  is a co-quasitriangular structure [8] on the outer Hopf algebra  $\Lambda$ , namely it fulfils the following properties

(i) Normalization:  $(\varepsilon \otimes \text{Id}) \circ r_\pi = \eta = (\text{Id} \otimes \varepsilon) \circ r_\pi$ ;

(ii) We have

$$\begin{aligned} (a) \quad (\text{Id} \otimes \Delta) \circ r_\pi &= r_\pi^{12} \cdot r_\pi^{13} \equiv (\text{m} \otimes \text{Id} \otimes \text{Id})(\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ r_\pi \otimes r_\pi; \\ (b) \quad (\Delta \otimes \text{Id}) \circ r_\pi &= r_\pi^{13} \cdot r_\pi^{23} \equiv (\text{Id} \otimes \text{Id} \otimes \text{m})(\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ r_\pi \otimes r_\pi. \end{aligned}$$

(iii) The antipode<sup>4</sup> relates  $r_\pi$  and its convolutive inverse  $r_\pi^{-1}$  as:

$$\begin{aligned} (\mathbf{S} \otimes \text{Id}) \circ r_\pi &= r_\pi^{-1} & (\text{Id} \otimes \mathbf{S}) \circ r_\pi^{-1} &= r_\pi \\ (\mathbf{S} \otimes \mathbf{S}) \circ r_\pi &= r_\pi & (\mathbf{S} \otimes \mathbf{S}) \circ r_\pi^{-1} &= r_\pi^{-1} \end{aligned}$$

**Proof:** For the normalization consider

$$(\varepsilon \otimes \text{Id}) \circ r_\pi = (\varepsilon \otimes \text{Id}) \sum (\alpha_{(1)}^\pi \otimes \alpha_{(2)}^\pi) = \sum \delta_{\alpha_{(1)}^\pi, (0)} \alpha_{(2)}^\pi = \eta, \quad (3-41)$$

where the penultimate term has to be interpreted in the light of (3-32). The tangle diagrams for conditions (ii)(a) and (ii)(b) read

$$\text{tangle diagrams} \quad (3-42)$$

<sup>4</sup>In  $\Lambda$  we have  $\mathbf{S}^2 = \text{Id}$  which simplifies the following relations.

To check that  $r_\pi^{-1}$  as given in the first part of (iii) is the inverse of  $r_\pi$  it suffices to note that

$$r_\pi^{-1} r_\pi = \sum_{\alpha, \beta} \alpha_{(1)}^\pi \beta_{(1)}^\pi \otimes \mathbf{S}(\alpha_{(2)}^\pi) \beta_{(2)}^\pi = \Delta'(L_\pi) \Delta'(M_\pi) = 1 = 1 \otimes 1. \quad (3-43)$$

The other cases are similar.  $\square$

**Corollary 3.9:**  $\Lambda$  under  $r_\pi$ : The outer Hopf algebra of symmetric functions  $\Lambda$  is a cobraided Hopf algebra. Dually,  $\Lambda$  with  $R_\pi$  as defined above is a braided Hopf algebra.  $R_\pi$  satisfies the Yang-Baxter relation

$$R_\pi^{12} R_\pi^{13} R_\pi^{23} = R_\pi^{23} R_\pi^{13} R_\pi^{12}. \quad (3-44)$$

■

**Proof:** This follows directly from Kassel [8] where the duality between quasitriangular and coquasitriangular structures is established. To emphasize the structure of the operators involved, however, we offer here a direct proof that the object  $c^\pi := \text{sw} \circ R_\pi$  is a braid. To shorten the proof we again adopt the short-hand notation  $r_\pi = \sum \alpha_{(1)}^\pi \otimes \alpha_{(2)}^\pi$  for the Sweedler sums occurring in the coscalar product, as well as Theorem 3.5 and  $R_\pi$ . Then we have

$$c^\pi(\lambda \otimes \mu) = \sum_{\alpha} \mu \cdot \prod_{k=1}^p \alpha_k[\pi'_{(1)}] \otimes \lambda \cdot \prod_{k=1}^p \alpha_k[\pi'_{(2)}] := \sum_{\alpha} \mu \cdot \alpha_{(1)}^\pi \otimes \lambda \cdot \alpha_{(2)}^\pi \quad (3-45)$$

We compute the left hand side of the braid equation (schematically written  $c^{12} c^{23} c^{12} = c^{23} c^{12} c^{23}$ ) as

$$\begin{aligned} (c^\pi \otimes \text{Id})(\text{Id} \otimes c^\pi)(c^\pi \otimes \text{Id})(\lambda \otimes \mu \otimes \nu) &= (c^\pi \otimes \text{Id})(\text{Id} \otimes c^\pi)(\mu \cdot \alpha_{(1)}^\pi \otimes \lambda \cdot \alpha_{(2)}^\pi \otimes \nu) \\ &= (c^\pi \otimes \text{Id})(\mu \cdot \alpha_{(1)}^\pi \otimes \nu \cdot \beta_{(1)}^\pi \otimes \lambda \cdot \alpha_{(2)}^\pi \cdot \beta_{(2)}^\pi) \\ &= (\nu \cdot \beta_{(1)}^\pi \cdot \gamma_{(1)}^\pi \otimes \mu \cdot \alpha_{(1)}^\pi \cdot \gamma_{(2)}^\pi \otimes \lambda \cdot \alpha_{(2)}^\pi \cdot \beta_{(2)}^\pi); \end{aligned}$$

the the right hand side is treated similarly,

$$\begin{aligned} (\text{Id} \otimes c^\pi)(c^\pi \otimes \text{Id})(\text{Id} \otimes c^\pi)(\lambda \otimes \mu \otimes \nu) &= (\text{Id} \otimes c^\pi)(c^\pi \otimes \text{Id})(\lambda \otimes \nu \cdot \alpha_{(1)}^\pi \otimes \mu \cdot \alpha_{(2)}^\pi) \\ &= (\text{Id} \otimes c^\pi)(\nu \cdot \alpha_{(1)}^\pi \cdot \beta_{(1)}^\pi \otimes \lambda \cdot \beta_{(2)}^\pi \otimes \mu \cdot \alpha_{(2)}^\pi) \\ &= (\nu \cdot \alpha_{(1)}^\pi \cdot \beta_{(1)}^\pi \otimes \mu \cdot \alpha_{(2)}^\pi \cdot \gamma_{(1)}^\pi \otimes \lambda \cdot \beta_{(2)}^\pi \cdot \gamma_{(2)}^\pi). \end{aligned}$$

Then with the re-labelling of the summations to interchange  $\alpha^\pi$  and  $\gamma^\pi$ , we have the required equality.  $\square$

## 4. KNOTS AND LINKS

**4.1. Knot alphabet.** In the introduction it was claimed that diagrammatic moves (tangle diagrams) in the character ring  $\Lambda$ , referred to as the **GL**-alphabet (1-3), could be used to assemble the ingredients for describing isotopy classes of knots via their two-dimensional projections, via a knot alphabet. In this section we develop this relationship in detail, using the additional operations on  $\Lambda$  afforded by its realisation as a formal character ring  $\text{Char-H}_\pi$ . As we shall see, the **GL**-moves become a degenerate case of the more general situation in  $\text{Char-H}_\pi$ . To evaluate the **Char-H}\_\pi** tangle we *rewrite* **Char-H}\_\pi** letters into **Char-GL** words. We shall refer to this as the transcription of the knot alphabet in terms of the  $\text{H}_\pi$ -alphabet. It is of paramount importance not to confuse the trivial, in the sense of knot theory, **GL**-tangles with the *nontrivial* transcription of the **Char-H}\_\pi** tangles into **Char-GL**



tangles. This gives us the following scheme (for the rewriting of crossings and similarly for caps, cups)

$$\begin{array}{ccc}
 \mathfrak{B}_n : \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & \xrightarrow{\text{trivialize}} & \mathfrak{S}_n : \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \\
 \text{hom} \downarrow & & \uparrow \text{trivialize } (r_\pi = \eta \otimes \eta) \\
 \text{Char-}\mathbf{H}_\pi : \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & \xleftrightarrow{\text{rewrite (iso)}} & \mathbf{GL} : \begin{array}{c} \text{cup} \quad \text{cap} \end{array}
 \end{array} \quad (4-46)$$

We now introduce the required ingredients step by step, via the tangle diagram and algebraic rewriting equivalents, defining the rewriting isomorphism. As mentioned, a major role is played by the 2-2 tangle crossing (braid)  $\mathbf{c}^\pi$  resulting from the  $\mathbf{GL}$ - $\mathbf{H}_\pi$  branching rule described in §3 above, and the left-right and right-left dualities, represented by caps and cups.

The main structural change is a passage to *oriented* tangles given by a formal distinction between  $\Lambda$  and its dual  $\Lambda^*$ . These are of course isomorphic as linear spaces and as Hopf algebras via the Schur-Hall scalar product, but the suite of moves entailed in the full  $\mathbf{H}_\pi$ -alphabet necessitates systematically accounting for combinations of operators on tensor products of spaces of both types. In fact this amounts to allowing co- and contravariant tensor characters and rational representations. As a matter of convention we define the following elementary 0-2 and 2-0 oriented tangles for the right-left duality:

$$\begin{array}{ll}
 \mathbf{b} \cong \begin{array}{c} \text{cup} \end{array} ; & \mathbf{d} \cong \begin{array}{c} \text{cup} \end{array} ; \\
 1 \mapsto \sum_{\sigma} \sigma \otimes \sigma^* ; & \lambda^* \otimes \mu \mapsto \langle \lambda | \mu \rangle,
 \end{array}$$

whose compositions implement the topological move **R0** on  $\Lambda$  and  $\Lambda^*$

$$\begin{array}{c} \text{cup} \end{array} \cong \begin{array}{c} \text{cup} \end{array}, \quad \begin{array}{c} \text{cup} \end{array} \cong \begin{array}{c} \text{cup} \end{array}. \quad (4-47)$$

Next we turn to the major ingredients of the knot alphabet, the braid  $\mathbf{c}^\pi$  and its inverse  $(\mathbf{c}^\pi)^{-1}$ . For these, the following over-under crossing representations are adopted to stand for the indicated tangle diagrams; the repositioning of  $r_\pi$  (for example as compared with the tangle diagram for  $R_\pi$  above) is easily checked:

**Definition 4.10:**  $\mathbf{H}_\pi$  alphabet structure of  $\mathbf{c}^\pi$  and  $(\mathbf{c}^\pi)^{-1}$ :

We have

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \Big|_{\mathbf{H}_\pi} \cong \begin{array}{c} \text{cup} \quad \text{cap} \end{array} \Big|_{\mathbf{GL}} \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \Big|_{\mathbf{H}_\pi} \cong \begin{array}{c} \text{cup} \quad \text{cap} \end{array} \Big|_{\mathbf{GL}} \quad (4-48)$$

where  $r_\pi^{-1}$  is the inverse of  $r_\pi$ , namely (see Theorem 3.8),

$$r_\pi = \sum \alpha_{(1)}^\pi \otimes \alpha_{(2)}^\pi, \quad r_\pi^{-1} = \sum S(\alpha_{(1)}^\pi) \otimes \alpha_{(2)}^\pi \quad (4-49)$$

■

Still missing from the compendium are the companion dual cap and cup diagrams  $\overline{b}_\pi$  and  $\overline{d}_\pi$ :

$$\overline{b}_\pi \cong \text{cap diagram} ; \quad \overline{d}_\pi \cong \text{cup diagram} ; \quad (4-50)$$

whose compositions implement the corresponding straightening rules

$$\text{wavy line} \cong \text{straight line} , \quad \text{straight line} \cong \text{wavy line} \quad (4-51)$$

As we now show, the  $\overline{b}_\pi$ ,  $\overline{d}_\pi$  moves must be *derived* from the foregoing ingredients and cannot be independently defined. Firstly, note that the following diagram representing an elementary twist, or formally, *writhe*, is represented by a linear operator which will play a crucial role:

**Definiton 4.11: Writhe operator:**

$$\text{twist diagram} \Leftrightarrow \lambda \mapsto \theta_\pi(\lambda). \quad (4-52)$$

■

The existence of the writhe operator  $\theta_\pi$  signifies that the knot diagrammar which we are constructing from the  $H_\pi$ -alphabet describes knot projections in ambient isotopy, that is including *framing* as part of the crossing information. Indeed, the removal of the twist represented by  $\theta_\pi$  is given by its inverse, implementing the modified (Reidemeister **R1'**) move (2-22),

$$\text{twist diagram} \cong \text{straight line} \cong \text{twist diagram} \quad (4-53)$$

Introducing a non-trivial writhe entails in the algebraic setting working with ribbon Hopf algebras.

We proceed to the dual cup and cap. We can rewrite these composed with untwisting moves, and rearranging, using only the assumed straightening operations for  $b$ ,  $d$ ,  $\overline{b}_\pi$  and  $\overline{d}_\pi$ , in order to derive alternative forms:

**Lemma 4.12: Dual cap  $\overline{b}_\pi$  and cup  $\overline{d}_\pi$ :** we have

$$\overline{b}_\pi := (\text{Id}_\Lambda \otimes (\theta_\pi)^{-1}) \circ c_{\Lambda\Lambda}^\pi \circ b, \quad \overline{d}_\pi := d \circ \overline{c}_{\Lambda\Lambda}^\pi \circ ((\theta_\pi)^{-1} \otimes \text{Id}_{\Lambda^*}). \quad (4-54)$$

■

**Proof:** Observe the equivalence of  $\overline{b}_\pi$  and  $\overline{d}_\pi$  to the following respective tangle diagrams which feature the original 0-2 and 2-0 tangles  $b$  and  $d$ , and evaluate:

$$\overline{b}_\pi : \text{cap diagram} := \text{tangle diagram} \quad \overline{d}_\pi : \text{cup diagram} := \text{tangle diagram} \quad (4-55)$$

□

The price paid for these rearrangements is the introduction of additional auxiliary crossings, related to  $c^\pi$ , which we now explain. Evidently, braidings on different spaces require labels such as  $c_{V,W}^\pi$ . Adopting this notation, the original braid  $c^\pi$  is here denoted  $c_{\Lambda,\Lambda}^\pi$  or  $c_{\Lambda\Lambda}^\pi$  for simplicity. Denoting right chirality by an overbar, for a choice of spaces  $\Lambda$ ,  $\Lambda^*$  on each line there are thus  $2^3 = 8$  types,

namely  $c_{\Lambda\Lambda}^\pi$  and  $\bar{c}_{\Lambda\Lambda}^\pi$  (the inverse  $(c_{\Lambda\Lambda}^\pi)^{-1}$ , as derived above), together with  $c_{\Lambda^*\Lambda}^\pi$  and  $\bar{c}_{\Lambda^*\Lambda}^\pi$ ,  $c_{\Lambda\Lambda^*}^\pi$  and  $\bar{c}_{\Lambda\Lambda^*}^\pi$ , as well as  $c_{\Lambda\Lambda^*}^\pi$  and  $\bar{c}_{\Lambda\Lambda^*}^\pi$ .

Dualisation and straightening of the basic crossings, together with appropriate identification of inverses, suffice for evaluation of all cases (including those needed to define  $\bar{b}_\pi$  and  $\bar{d}_\pi$ ). The following lemma provides a complete dictionary of these crossings (including, for completeness, the standard ones), both in diagrammatic form, and as well as giving their explicit action on elements of the appropriate space  $\Lambda^{(*)} \otimes \Lambda^{(*)}$ , in the  $S$ -function basis:

**Lemma 4.13: Knot to Char- $H_\pi$  dictionary – I. Crossings:** In addition to the standard crossings listed, the table 1 defines the auxiliary crossings in terms of oriented over- and under-crossings, in terms of primitive tangles involving operations in Char-GL, and in terms of their action on characters belonging to the relevant spaces, see table 1. ■

**Proof:** In table 1 we construct the six auxiliary crossings for partly upward oriented lines representing elements of  $\Lambda^*$ , and we provide the rewriting homomorphism Char- $H_\pi$  to Char-GL for all crossings. The left column gives the name of the crossing, the Char- $H_\pi$  column provides the tangle version including orientation, and further shows how the auxiliary crossing is obtained by using the caps and cups. Hence the crossing appears as either  $c_{UV}^\pi$  or  $\bar{c}_{UV}^\pi$  with  $U, V \in \{\Lambda, \Lambda^*\}$ . The first four auxiliary crossings use composition with  $b$  and  $d$ . The last two cannot be obtained this way, as we would need  $\bar{b}_\pi$  and  $\bar{d}_\pi$  tangles, which we define from the  $b, d$  closed structure and the crossing. However, these two cases can be obtained as inverse tangles under vertical composition with  $\bar{c}_{\Lambda^*\Lambda}^\pi$  and  $c_{\Lambda^*\Lambda}^\pi$ . The fact that we can produce inverses under both vertical and horizontal composition relies upon the identity  $r_\pi^{-1} r_\pi = 1$ . The Char-GL tangles provide the image of the crossings in Char-GL, and hence the GL-word which faithfully represents the crossing. The last column gives the algebraic expression of the crossings.

For a proof we present the algebraic identity, showing how the switch is modified by the insertion of terms  $\alpha^\pi$  or  $S(\alpha^\pi)$  coming from the crossing. The main thing to note is that on  $\Lambda$  one acts by Schur function multiplication, while on  $\Lambda^*$  one acts by skewing. Following the fate of such summands in the sliced tangle diagram for one case, we have

$$\begin{aligned}
c_{\Lambda^*\Lambda}^\pi : \lambda^* \otimes \mu &\mapsto \sum_{\sigma} \lambda^* \otimes \mu \otimes \sigma \otimes \sigma^* \\
&\mapsto \sum_{\sigma, \alpha} \lambda^* \otimes \sigma \cdot \alpha_{(1)}^\pi \otimes \mu \cdot \alpha_{(2)}^\pi \otimes \sigma^* \\
&\mapsto \sum_{\sigma, \alpha} \langle \lambda | \sigma \cdot \alpha_{(1)}^\pi \rangle \mu \cdot \alpha_{(2)}^\pi \otimes \sigma^* \\
&= \sum_{\sigma, \alpha} \langle \lambda / \alpha_{(1)}^\pi | \sigma \rangle \mu \cdot \alpha_{(2)}^\pi \otimes \sigma^* \\
&= \sum_{\alpha} \mu \cdot \alpha_{(2)}^\pi \otimes (\lambda / \alpha_{(1)}^\pi)^* \tag{4-56}
\end{aligned}$$

with the sum over  $\sigma$  enforcing  $\lambda^* \mapsto (\lambda / \alpha^\pi)^*$  on each dual line. In this way, the Sweedler parts of  $r_\pi$  involved in the crossing systematically appear with outer skew rather than outer product when they appear with elements of  $\Lambda^*$ , while the crossing handedness is still reflected in the presence of  $(r_\pi)^{-1}$  via the antipode. □

TABLE 1. Crossings, Char- $H_\pi$ , Char-GL tangles and algebraic forms of the 8 possible forms of oriented crossings.

braid	Char- $H_\pi$ tangle	Char-GL tangle	algebraic expression
$c_{\Lambda\Lambda}^\pi$			$\lambda \otimes \mu \mapsto \sum_\alpha \mu \cdot \alpha_{(2)}^\pi \otimes \lambda \cdot \alpha_{(1)}^\pi$
$\bar{c}_{\Lambda\Lambda}^\pi$			$\lambda \otimes \mu \mapsto \sum_\alpha \mu \cdot S(\alpha_{(2)}^\pi) \otimes \lambda \cdot \alpha_{(1)}^\pi$
$c_{\Lambda^*\Lambda}^\pi$			$\lambda^* \otimes \mu \mapsto \sum_\alpha \mu \cdot \alpha_{(2)}^\pi \otimes (\lambda/\alpha_{(1)}^\pi)^*$
$\bar{c}_{\Lambda^*\Lambda}^\pi$			$\lambda^* \otimes \mu \mapsto \sum_\alpha \mu \cdot S(\alpha_{(2)}^\pi) \otimes (\lambda/\alpha_{(1)}^\pi)^*$
$c_{\Lambda^*\Lambda^*}^\pi$			$\lambda^* \otimes \mu^* \mapsto \sum_\alpha (\mu/\alpha_{(2)}^\pi)^* \otimes (\lambda/\alpha_{(1)}^\pi)^*$
$\bar{c}_{\Lambda^*\Lambda^*}^\pi$			$\lambda^* \otimes \mu^* \mapsto \sum_\alpha (\mu/S(\alpha_{(2)}^\pi))^* \otimes (\lambda/\alpha_{(1)}^\pi)^*$
$c_{\Lambda\Lambda^*}^\pi$			$\lambda \otimes \mu^* \mapsto \sum_\alpha (\mu/\alpha_{(2)}^\pi)^* \otimes \lambda \cdot \alpha_{(1)}^\pi$
$\bar{c}_{\Lambda\Lambda^*}^\pi$			$\lambda \otimes \mu^* \mapsto \sum_\alpha (\mu/S(\alpha_{(2)}^\pi))^* \otimes \lambda \cdot \alpha_{(1)}^\pi$

With the above list of crossings in hand, we turn to the evaluation of explicit forms for  $\bar{b}_\pi$ ,  $\bar{d}_\pi$  and  $\theta_\pi$  to complete the transcription of the knot alphabet in Char- $H_\pi$  into moves in Char-GL. We have:

**Lemma 4.14: Knot to Char- $H_\pi$  dictionary – II:** In addition to the basic crossings defined in Lemma 4.13 above, the following elements of the Char- $H_\pi$ -knot alphabet are defined diagrammatically and by their action on states, see table 2: ■

**Proof:**

As already noted, the equivalence (lemma 4.12) of  $\bar{b}_\pi$  and  $\bar{d}_\pi$  to tangle diagrams which feature the original 0-2 and 2-0 tangles  $b$  and  $d$  can now be exploited in the light of the explicit forms for the

TABLE 2. left-right cups, caps, twists, as Char- $H_\pi$ , Char-GL tangles and their algebraic forms

map	Char- $H_\pi$ tangle	Char-GL tangle	algebraic expression
b			$1 \mapsto \sum \sigma \otimes \sigma^*$
d			$\lambda^* \otimes \mu \mapsto \langle \lambda   \mu \rangle$
$\bar{b}_\pi$			$1 \mapsto \sum \rho^* \otimes \rho$
$\bar{d}_\pi$			$\lambda \otimes \mu^* \mapsto \langle \mu   \lambda \rangle$
$\theta_\pi$			$\lambda \mapsto Q_\pi \cdot \lambda$
$(\theta_\pi)^{-1}$			$\lambda \mapsto (Q_\pi)^{-1} \cdot \lambda$

auxiliary crossings given above (lemma 4.13). Taking for example the expression for  $\bar{b}_\pi$ , we have for its action in sliced form

$$\begin{aligned}
 1 &\mapsto \sum \sigma \otimes \sigma^* \mapsto \sum (\sigma / \alpha_{(1)}^\pi)^* \otimes \sigma \cdot \alpha_{(2)}^\pi \\
 &\mapsto \sum (\sigma / \alpha_{(1)}^\pi)^* \otimes (\theta_\pi)^{-1} (\sigma \cdot \alpha_{(2)}^\pi).
 \end{aligned} \tag{4-57}$$

However as a consequence of outer product associativity,  $\sum_\sigma \sigma / \alpha \otimes \sigma \cdot \beta = \sum_\sigma \sigma \otimes \sigma \cdot (\alpha \cdot \beta)$  and for the Sweedler sums entailed in  $r_\pi$  this just produces the Cauchy scalar element  $Q_\pi$  on the right-hand side, namely we infer  $\bar{b}_\pi(1) = \sum \sigma^* \otimes (\theta_\pi)^{-1} (Q_\pi \cdot \sigma)$ . On the other hand, the expression for  $\bar{b}_\pi$  could have been written using the inverse Reidemeister move instead. Using this alternative, and following the same progression through the sliced diagram now leads to  $\bar{b}_\pi(1) = \sum \sigma^* \otimes \theta_\pi ((Q_\pi)^{-1} \cdot \sigma)$ . Equating these requires  $\theta_\pi ((Q_\pi)^{-1} \cdot \sigma) = (\theta_\pi)^{-1} (Q_\pi \cdot \sigma)$  or  $(\theta_\pi)^2(\rho) = (Q_\pi)^2 \cdot \rho$  for arbitrary  $\rho$ . We solve this functional equation by simply *identifying* the linear operator  $\theta_\pi$  with the multiplication by  $Q_\pi$ . This immediately means that  $\bar{b}_\pi$ , and similarly  $\bar{d}_\pi$ , collapse to their minimal forms, with  $\theta_\pi$  corresponding to multiplication by  $Q_\pi$ , as claimed.

Alternatively, the elementary twist itself can be analysed via the algebraic steps given in the corresponding sliced tangle diagram of  $\overline{\mathbf{b}}_\pi$ , and we find a consistent result, namely

$$\begin{aligned}
 \text{Diagram} &\Leftrightarrow \lambda \mapsto \sum \lambda \otimes \sigma \otimes \sigma^* \\
 &\mapsto \sum \sigma \cdot \alpha_{(1)}^\pi \otimes \lambda \cdot \alpha_{(2)}^\pi \otimes \sigma^* \\
 &\mapsto \sum \sigma \cdot \alpha_{(1)}^\pi \sigma^* (\lambda \cdot \alpha_{(2)}^\pi) \equiv \sum \alpha_{(1)}^\pi \cdot \left( \sum_\sigma \sigma \langle \sigma | \lambda \cdot \alpha_{(2)}^\pi \rangle \right) \\
 &= \lambda \cdot \sum (\alpha_{(1)}^\pi \cdot \alpha_{(2)}^\pi) \equiv Q_\pi \cdot \lambda
 \end{aligned} \tag{4-58}$$

as required. In the second and third lines the forms of the standard cap- and dual cup- operators  $\mathbf{b}$  and  $\overline{\mathbf{d}}_\pi$  have been implemented, together with the resolution of the identity as  $\text{Id} = \sum_\sigma \sigma \circ \sigma^*$  in an orthonormal basis). As a consistency check, if in lemma 4.13 above, the three crossings derived as inverses are instead evaluated in terms of standard crossings, using ‘unstraightening’ moves involving  $\overline{\mathbf{b}}_\pi$  and  $\overline{\mathbf{d}}_\pi$ , the same expressions result.

Finally we need to check that the alternative versions of the  $\overline{\mathbf{b}}_\pi, \overline{\mathbf{d}}_\pi$  and twist tangles yield the same translation to ensure that our map is indeed a homomorphism. This can be done by a tangle argument for all cases in the table 2.  $\square$

The last two lemmas 4.13 and 4.14 can be summarized as

**Proposition 4.15:** The *rewrite* map from the closed braid part of  $\text{Char-H}_\pi$  into  $\text{Char-GL}$  given in 4-46 is an isomorphism of groups.  $\blacksquare$

Equipped with this structure we finally obtain the

**Theorem 4.16:  $\Lambda$  as a ribbon Hopf algebra:** For each partition  $\pi$  the space  $\Lambda \cong \text{Char-H}_\pi$  equipped with the braid operators  $\mathbf{c}^\pi, (\mathbf{c}^\pi)^{-1}$  together with the objects  $\mathbf{b}, \mathbf{d}, \overline{\mathbf{b}}_\pi, \overline{\mathbf{d}}_\pi$  and the canonical writhe element  $Q_\pi$ , is a ribbon Hopf algebra [8].  $\blacksquare$

**4.2. Knot invariant operators.** The identification of all the ingredients of the knot alphabet in terms of moves in  $\text{Char-H}_\pi$  confers on it a status equivalent to that of  $\text{Char-GL}$  itself. Note that the presentation of tangle relations in  $\text{Char-GL}$  (see Figure 2-21 following Theorem 2.1, §2), contains several implicit steps such as the introduction of cup and cap operators and their duals. In the  $\text{Char-H}_\pi$  case, by contrast, these have had to be considered in detail. Although the final forms for  $\mathbf{b}, \mathbf{d}, \overline{\mathbf{b}}_\pi$  and  $\overline{\mathbf{d}}_\pi$  are indeed equivalent to those for  $\text{Char-GL}$  (adapted to the case of directed tangles), it should be emphasised that these definitions are by no means automatic.

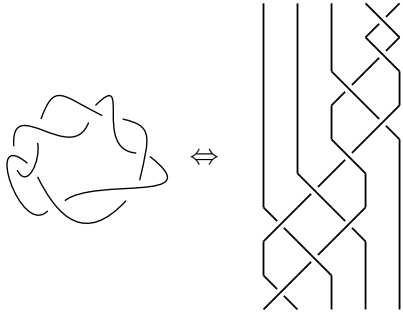
We finally return to the discussion of the introduction to the paper, §1, where it was mentioned that one aim of our current exploration of manipulations of group characters rings was the possibility of generating new knot invariants. With the full apparatus of ribbon Hopf algebras in place, we can now apply the well-established formalism (see §1) whereby tangle diagrams can be associated with knot projections, with their evaluations in  $\Lambda$  and its tensor powers guaranteed to be invariants.

Complete knots and links are projected as decorated images of products of circles, and so must be interpreted in terms of slicings of 0-0 tangles. Consider the oriented unknots, and their evaluation according to the obvious slicings; for example (in the left-handed case)

$$\begin{aligned}
 \text{Diagram} &\Leftrightarrow 1 \mapsto \\
 &\mapsto \sum_\sigma \sigma \otimes \sigma^* \\
 &\mapsto \sum_\sigma \langle \sigma | \sigma \rangle \equiv \sum_\sigma 1,
 \end{aligned} \tag{4-59}$$

a formally infinite sum. It is clear however that a finite result can be arranged for this and other links and knots by simply cutting and opening one strand (or more) of the 0-0 tangle, and evaluating the resulting 1-1 tangle invariant as an element of  $End(\Lambda)$  (or of  $End(\otimes^k \Lambda)$  for a  $k$ - $k$  tangle). Clearly, for the left- and right-handed unknots, this simply produces the identity operator on  $\Lambda$  or  $\Lambda^*$ , respectively, but is potentially more interesting for general knots and links.

Consider then a general link projection, made into a 1-1 tangle as described. By standard theorems it is regular-isotopic to the corresponding single-strand opening of a braid word representation of the knot or link, and it is in this canonical form that we proceed with an evaluation. Thus suppose that the knot  $K$  is isotopic to the closure of a braid element in  $\mathfrak{B}_m$ , by means of a word of length  $\ell$ ,  $b_K = b_{i_1}^{e_1} b_{i_2}^{e_2} \cdots b_{i_\ell}^{e_\ell}$ , where each  $i_k \in \{1, 2, \dots, m-1\}$ , with each exponent  $e_i = \pm 1$ . Clearly  $\ell \equiv \sum_i |e_i|$ , while the sum  $w = \sum_i e_i := w_+ - w_-$  is the writhe of the knot or link projection (the difference between the positive and negative exponent sums).



The knot 8\_1 and its braid representation. (4-60)

The situation is as illustrated in the figure 4-60, which shows the specific case of the knot 8\_1 and a braid representation of it<sup>5</sup>. Writhe is introduced under the map from the knot to the braid provided by the Alexander theorem but they always appear in mutually inverse pairs on the same line.

In order to write down the operator in  $\Lambda$  depicted by the sliced diagram for such a braid tangle, using the above-developed toolkit in  $\text{Char-H}_\pi$ , it is only necessary to use the rules for the basic crossing  $c_\pi$  and its inverse. Schematically, the representation of the knot as a braid closure in  $\mathfrak{B}_m$  means that there will be a nested sequence of  $m-1$  cap tangles, and corresponding cups, together with one open line, say the first, representing the cut strand. Assuming that the (downward) strands are labelled  $\sigma_1, \sigma_2, \dots, \sigma_m$ , with  $\sigma_1$  the character label on the open line, the evaluation amounts to working out the action of the knot invariant operator  $\mathcal{I}_K : \sigma_1 \mapsto \mathcal{I}_K \sigma_1$ . Starting with  $\sigma_1$ , the final result of applying the nested sequence of caps, braid crossings and then cups will give for  $\mathcal{I}_K$  a product of the form

$$\mathcal{I}_K \sigma_1 = \sum \sigma_{\kappa_1} \cdot a_1 \langle \sigma_{\kappa_2} \cdot a_2 | \sigma_2 \rangle \langle \sigma_{\kappa_3} \cdot a_3 | \sigma_3 \rangle \cdots \langle \sigma_{\kappa_m} \cdot a_m | \sigma_m \rangle. \quad (4-61)$$

Here  $\kappa \in \mathfrak{S}_m$  is the image of  $b_K$  under the projection  $\mathfrak{B}_m \rightarrow \mathfrak{S}_m$  reflecting the permutation on strands induced by the braiding and the original labelling of strands induced by the nested caps. In implementing the summations entailed in  $c_\pi$ ,  $c_\pi^{-1}$ , there will be  $\ell$  sets of additional summations  $\alpha_{(\pi)}^1, \alpha_{(\pi)}^2, \dots, \alpha_{(\pi)}^\ell$  symbolising the composite sums involved in  $r_\pi$ ; the summands appear repeated on correlated strands involved in the various crossings. Moreover, for  $w_-$  of these pairs, one of these occurrences will entail the antipode  $S(\cdot)$  reflecting the inverse crossings. The objects  $a_i$  accompanying each  $\sigma_{\kappa_i}$  thus represent the distribution of products of these summands resulting from the application of the crossings.

<sup>5</sup>Taken from the Rolfsen knot table, [http://katlas.org/wiki/The\\_Rolfsen\\_Knot\\_Table](http://katlas.org/wiki/The_Rolfsen_Knot_Table)

In the case of the braid representation of knot 8<sub>1</sub> given above, however, all crossings are positive and the braidings lead to

$$\sigma_4 \cdot a_1 \otimes \sigma_5 \cdot a_2 \otimes \sigma_1 \cdot a_3 \otimes \sigma_2 \cdot a_4 \otimes \sigma_3 \cdot a_5 \otimes \sigma_2 \otimes \cdots \otimes \sigma_5, \quad (4-62)$$

with  $a_1 = \alpha_1 \alpha_2 \alpha_7 \alpha_9 \alpha_{10}$ ,  $a_2 = \alpha_1 \alpha_2 \alpha_4 \alpha_5 \alpha_6 \alpha_8$ ,  $a_3 = \alpha_8 \alpha_9$ ,  $a_4 = \alpha_6 \alpha_7$ , and  $a_5 = \alpha_3 \alpha_4$ , with the  $\alpha \cdots \alpha$  standing for the  $\alpha_{(1)(\pi)} \cdots \alpha_{(2)(\pi)}$  summations over Sweedler part plethysms associated with the 10 crossings in this case. Including the bottom cups corresponding to the braid closures (on all but the first lines) indeed leads to an expression of the form given in (4-61) above, where the permutation  $\kappa \in \mathfrak{S}_5$  is the 5-cycle  $\kappa = (4, 5, 1, 2, 3)$ , or (42531) in cycle notation.

We pursue the general analysis for the case that  $\kappa$  is an  $m$ -cycle,  $\kappa = (1\kappa_1\kappa_2 \cdots \kappa_{m-1})$  in cycle notation. The strand  $\sigma_1$  occurs in an outer product with  $a_{\kappa^{-1}1}$  where  $\kappa^{-1}1 = \kappa_{m-1}$ ; this product is paired in a scalar product with the accompanying  $\sigma_{\kappa_{m-1}}$ , which in turn occurs in a similar arrangement with  $a_{\kappa_{m-2}}$  and  $\sigma_{\kappa_{m-2}}$ . Fixing on the sum over  $\sigma_{\kappa_{m-1}}$  and denoting the remaining entries by  $\cdots$ , we have

$$\sum \cdots \langle \sigma_1 \cdot a_{\kappa_{m-1}} | \sigma_{\kappa_{m-1}} \rangle \langle \sigma_{\kappa_{m-1}} \cdot a_{\kappa_{m-2}} | \sigma_{\kappa_{m-2}} \rangle \equiv \sum \cdots \langle \sigma_1 \cdot a_{\kappa_{m-1}} a_{\kappa_{m-2}} | \sigma_{\kappa_{m-2}} \rangle, \quad (4-63)$$

using of the resolution of the identity in the sum over  $\sigma_{\kappa_{m-1}}$ . Clearly this process can be iterated, leaving the last sum over  $\sigma_{\kappa_1}$  in the form

$$\sum \sigma_{\kappa_1} \cdot a_1 \langle \sigma_1 \cdot a_{\kappa_{m-1}} a_{\kappa_{m-2}} \cdots a_{\kappa_1} | \sigma_{\kappa_1} \rangle. \quad (4-64)$$

which yields

$$\mathcal{I}_K \sigma_1 = \sum \sigma_1 \cdot a_{\kappa_{m-1}} a_{\kappa_{m-2}} \cdots a_{\kappa_1}, \quad (4-65)$$

and (the outer product being commutative) we recover the factor  $\prod_{i=1}^{\ell} a_i$ . This contains the products of *all*  $\ell$  paired sets of  $\alpha_{(\pi)}$  summands, including the terms which carry an antipode associated with inverse crossings. Clearly then, as the product is commutative, these factors can be rearranged as  $w_+$  powers of  $Q_\pi$ , and  $w_-$  powers of  $Q_\pi^{-1}$ , the result being simply  $\mathcal{I}_K = (Q_\pi)^w$  as a multiplicative factor.

If  $\kappa$  is *not* an  $m$ -cycle, then there are say  $k > 1$  sets of strands (corresponding to the cycle structure of  $\kappa$ ) which do not communicate: the braid represents a link projection with two or more  $S^1$  components. In this case a similar argument to the above goes through if the invariant is evaluated for a cut diagram wherein one representative strand from *each* cycle type is cut, giving an link projection invariant operator  $\mathcal{I}_L$  on  $\otimes^k \Lambda$ . For example, if  $k = 2$  and the linked knots are  $K_1$  and  $K_2$ , the evaluation leads to

$$\mathcal{I}_L = (Q_\pi)^{w_1} \otimes (Q_\pi)^{w_2} \cdot (r_\pi)^{w_{12}} \quad (4-66)$$

where  $w_{12}$  is the *linking number* of the two knots.

## REFERENCES

- [1] Hélène Barcelo and Arun Ram. *Combinatorial representation theory*, pages 23–90. Cambridge Univ. Press, Cambridge, Berkeley, CA, 1999. Math. Sci. Res. Inst. Publ., 38.
- [2] Christian Brouder, Bertfried Fauser, Alessandra Frabetti, and Robert Oeckl. Quantum field theory and Hopf algebra cohomology [formerly: Let's twist again]. *J. Phys. A.*, 37:5895–5927, 2004. hep-th/0311253.
- [3] Bertfried Fauser. A Treatise on Quantum Clifford Algebras. Konstanz, 2002. Habilitationsschrift, arXiv:math.QA/0202059.
- [4] Bertfried Fauser and P. D. Jarvis. A Hopf laboratory for symmetric functions. *J. Phys. A: Math. Gen.*, 37(5):1633–1663, 2004. math-ph/0308043.



- [5] Bertfried Fauser, Peter D. Jarvis, Ron C. King, and Brian G. Wybourne. New branching rules induced by plethysm. *J. Phys A: Math. Gen.*, 39:2611–2655, 2006. math-ph/0505037, available from IoP Select <http://www.iop.org/Select/>.
- [6] Bertfried Fauser, Ronald C. King, and Peter D. Jarvis. A Hopf algebraic approach to the theory of group branchings. In R.C. King, M. Bylicki, and J. Karwowski, editors, *Symmetry, Spectroscopy, and SCHUR, Proceedings of the Prof. Brian G. Wybourne Commemorative Meeting, Toruń, 12.-14. June 2005*, pages 1–16, Toruń, 2006. N. Copernicus Univ. Press.
- [7] Bertfried Fauser, P. D. Jarvis, and Ronald C. King. Hopf algebra structure of the character rings of orthogonal and symplectic groups. arXiv:0704.2029v1 preprint, 34pp, 2007.
- [8] Christian Kassel. *Quantum groups*. Springer-Verlag, New-York - Berlin - Heidelberg, 1995.
- [9] Louis H. Kauffman. *Knots and physics*. World Scientific, Singapore, 1991.
- [10] Greg Kuperberg. Involutory Hopf algebras and 3-manifold invariants. *International Journal of Mathematics*, 2(1):41–66, 1991.
- [11] D. E. Littlewood. *The Theory of Group Characters*. Oxford University Press, Oxford, 1940. AMS Chelsea Publishing, reprint of the 1976 Dover ed., 2006.
- [12] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Calderon Press, Oxford, 1979. [2nd edition 1995].
- [13] M. J. Newell. Modification laws for the orthogonal and symplectic groups. *Proc. Roy. Soc. Irish Acad.*, (54):153–156, 1951.
- [14] Tomotada Ohtsuki. *Quantum Invariants*. World Scientific, Singapore, 2002. Series on Knots and Everything - Vol. 29.
- [15] Zbigniew Oziewicz, talk at ICCA5, in Ixtapa, Mexico, 1999.
- [16] Robert A. Proctor. Odd symplectic groups. *Invent. Math.*, 92:307–332, 1988.
- [17] Gian-Carlo Rota and Joel A. Stein. Plethystic algebras and vector symmetric functions. *Proc. Natl. Acad. Sci. USA*, 91:13062–13066, December 1994.
- [18] Gian-Carlo Rota and Joel A. Stein. Plethystic Hopf algebras. *Proc. Natl. Acad. Sci. USA*, 91:13057–13061, December 1994.
- [19] Moss E. Sweedler. *Hopf Algebras*. W. A. Benjamin, INC., New York, 1969.
- [20] Moss Eisenberg Sweedler. Cohomology of algebras over Hopf algebras. *Trans. Am. Math. Soc.*, 133:205–239, 1968.
- [21] Jean-Yves Thibon. Coproduits de fonctions symétrique. *C. R. Acad. Sci. Paris, t. 312, Série I*, pages 553–556, 1991.
- [22] Jean-Yves Thibon. Hopf algebras of symmetric functions and tensor products of symmetric group representations. *International Journal of Algebra and Computation*, 1(2):2007–221, 1991.
- [23] Vladimir G. Turaev. *Quantum invariants of knots and 3-manifolds*. De Gruyter, Berlin; New York, 1994.

SCHOOL OF COMPUTER SCIENCE, THE UNIVERSITY OF BIRMINGHAM, EDGBASTON-BIRMINGHAM, W. MIDLANDS, B15 2TT, ENGLAND

*E-mail address:* fauser.b@cs.bham.ac.uk

SCHOOL OF MATHEMATICS AND PHYSICS, UNIVERSITY OF TASMANIA, PRIVATE BAG 37, GPO, HOBART TAS 7001, AUSTRALIA

*E-mail address:* Peter.Jarvis@utas.edu.au

SCHOOL OF MATHEMATICS, UNIVERSITY OF SOUTHAMPTON, SOUTHAMPTON SO17 1BJ, ENGLAND

*E-mail address:* R.C.King@soton.ac.uk